S matrix of collective field theory

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By applying the Lehmann-Symanzik-Zimmermann (LSZ) reduction formalism, we study the S matrix of collective field theory in which fermi energy is larger than the height of potential. We consider the spatially symmetric and antisymmetric boundary conditions. The difference is that S matrices are proportional to momenta of external particles in antisymmetric boundary condition, while they are proportional to energies in symmetric boundary condition. To the order of g_{st}^2, we find simple formulas for the S matrix of general potential. As an application, we calculate the S matrix of a case which has been conjectured to describe a "naked singularity".

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I. INTRODUCTION

The one-dimensional hermitian matrix model can be studied by fermionic method and also by bosonic collective field theory. In the double scaling limit, the fermionic methods are indeed powerful to give any scattering amplitude of fermion density perturbation in terms of integrals of reflection coefficients[1,2]. For some cases, as a computational tool, the fermionic method may be much more efficient than the bosonic one. As will be argued later, however, the bosonic method[3] has its own right when complicated potential is considered.

In Ref.[4], through a beautiful interpretation of an excitation of fermi surface, the general form of classical collective field was given, and it was used to make relation between incoming and outgoing waves which gives the S matrix. The relation was completely solved to obtain the classical S matrix[5]. The study of the quantum collective field theory has been carried out[6,7], and the results reasonably agree with those from fermionic method or classical method[1,2,4,5].

In the double-scaled quantum collective field theory of fermi energy smaller than the height of potential (the $\mu > 0$ case), divergences appears due to the classical turning point[6], while there is no such divergence for the $\mu < 0$ case[8,7]. In the $\mu < 0$ case, the "time of flight" τ could be defined in $(-\infty, \infty)$ and the theory is described by a massless (Tachyon) field $X(t, \tau)$ in two-dimensional Minkowski spacetime. Since two-dimensional massless field X is unphysical and is a fluctuations around some background, it may be necessary to consider only the half of degree of freedom, as shown in

the Liouville theory[9]. This can be done by imposing the spatially symmetric boundary condition ($X(t, \tau) = X(t, -\tau)$) or spatially antisymmetric boundary condition ($X(t, \tau) = -X(t, -\tau)$). Since the quantum states recognize global information, in general quantum theory could be sensitive to the boundary conditions[10].

In this paper, we will study the quantum S matrix of double-scaled $\mu < 0$ collective field theory of *arbitrary potential*, by applying the LSZ reduction method[11]. Both of the boundary conditions will be considered. For the symmetric boundary condition it will be shown that S matrix can be written in terms of energies ($|p_i|$) of external particles, while S matrix are proportional to the momentum (p_i) in the antisymmetric boundary condition. For the usual inverted harmonic oscillator potential, our results of symmetric boundary condition are basically same to those of Ref.[7] except factors: square root of product of external energies which are needed for the correct aymptotics. However, a step further has been done in this paper; By making use of convolution theorem, the S matrix of order of g_{st}^2 are written in terms of velocity, which could simplify the analyses. As an application, the S matrix of the potential which has been conjectured to describe a "naked singularity" are obtained[12,13].

In the next section, we introduce Hamiltonian and boundary condition. A brief argument of applicability of LSZ reduction method in our case will be included, since there has been no discussion in previous publications. Sec. III and Sec.IV will be devoted to find the S matrices in symmetric and antisymmetric boundary conditions, respectively. Applications of the

general formulas will be made in Sec.V. We conclude in Sec.VI with some remarks.

II. HAMILTONIAN AND BOUNDARY CONDITION

The Hamiltonian of $\mu < 0$ collective field theory may be written as[7,8]

$$\mathcal{H} = -\frac{1}{4}\int_{-\infty}^{\infty} d\tau : [P^2 + X'^2 - \frac{\sqrt{\pi}}{\beta v^2}(PX'P + \frac{1}{3}X'^3) - \frac{1}{2\beta\sqrt{\pi}}(\frac{v''}{3v^3} - \frac{(v')^2}{2v^4})X'] :, \tag{1}$$

where the primes denote derivatives with respect to τ and the spatial coordinate τ is related to the eigenvalue λ of matrix model by

$$v(\lambda) = \frac{d\lambda}{d\tau} = \sqrt{2(\mu_F - U(\lambda))}. \tag{2}$$

In the double-scaled theory, the velocity v is expected to be proportional to $\sqrt{|\mu|} = \sqrt{|\mu_c - \mu_F|}$ where μ_F and μ_c are fermi energy and height of the potential $U(\lambda)$, respectively. And the coupling constant of the theory is $g_{st} = \frac{1}{\beta|\mu|}$. For convenience, we define $f_1(\tau),\ f_3(\tau)$ as

$$\begin{aligned} f_1(\tau) &= \mid \mu \mid (\frac{v''}{3v^3} - \frac{(v')^2}{2v^4}), && (3)\\ f_3(\tau) &= \mid \mu \mid \frac{1}{v^2(\tau)}, && (4) \end{aligned}$$

and their Fourier transforms as

$$g_a(k) = \int_{-\infty}^{\infty} e^{ik\tau} f_a(\tau) d\tau \qquad (\text{for} \quad \text{any} \quad a). \tag{5}$$

A very useful property of $\mathcal{H}$ is that[14,4]

$$\mathcal{H} = \frac{1}{2}(\mathcal{H}_R + \mathcal{H}_L), \tag{6}$$

where

$$\mathcal{H}_R = \frac{1}{4}\int_{-\infty}^{\infty} d\tau : [(P-X')^2 + \frac{\sqrt{\pi}}{3} g_{st} f_3(\tau)(P-X')^3 + \frac{g_{st}}{2\sqrt{\pi}} f_1(\tau)(P-X')] :, \quad (7)$$

$$\mathcal{H}_L = \frac{1}{4}\int_{-\infty}^{\infty} d\tau : [(P+X')^2 - \frac{\sqrt{\pi}}{3} g_{st} f_3(\tau)(P+X')^3 - \frac{g_{st}}{2\sqrt{\pi}} f_1(\tau)(P+X')] : . \quad (8)$$

A two-dimensional massless field (tachyon) may be written as

$$X(t,\tau) = \int_{-\infty}^{\infty} \frac{dk}{\sqrt{4\pi \mid k \mid}} [a(k) e^{ik\tau - i|k|t} + a^\dagger(k) e^{-ik\tau + i|k|t}] \quad (9)$$

with hermitian operators $a,\ a^\dagger$ satisfying

$$[a(k), a^\dagger(k')] = \delta(k-k'), \qquad [a(k), a(k')] = 0. \quad (10)$$

If we assume the symmetric boundary condition for X, the operator $a(k)$ satisfy $a(k) = a(-k)$, which gives the relation

$$(p+X')(t,\tau) = (P-X')(t,-\tau).$$

Therefore, if we impose the symmetric boundary condition on X, the interaction term of $\mathcal{H}$ vanish and it becomes free theory.

In the antisymmetric boundary condition, the operator $a(k)$ satisfy $a(k) = -a(-k)$ so that $(p+X')(t,\tau) = -(P-X')(t,-\tau)$. In this boundary condition, therefore,

$$\mathcal{H} = \mathcal{H}_R = \mathcal{H}_L \quad (11)$$

or

$$\begin{aligned} \mathcal{H} &= \int_0^\infty dkk : a^\dagger(k) a(k) : \\ &+ \frac{ig_{st}}{12\pi} \int_0^\infty dk_1 dk_2 dk_3 \sqrt{k_1 k_2 k_3} [g_3(k_1+k_2+k_3) : a(k_1) a(k_2) a(k_3) : \end{aligned}$$

$$
\begin{aligned}
&- 3g_3(k_1+k_2-k_3) : a(k_1)a(k_2)a^\dagger(k_3) : \\
&+ 3g_3(-k_1-k_2+k_3) : a^\dagger(k_1)a^\dagger(k_2)a(k_3) : \\
&- g_3(-k_1-k_2-k_3) : a^\dagger(k_1)a^\dagger(k_2)a^\dagger(k_3) :] \\
-\frac{ig_{st}}{8\pi}\int_0^\infty dk\sqrt{k}(g_1(k)a(k) - g_1(-k)a^\dagger(k)),
\end{aligned}
\tag{12}
$$

which will be considered as the Hamiltonian of our system afterwards. A useful expression of $\mathcal{H}$ for practical calculation is

$$
\begin{aligned}
\mathcal{H} &= -\frac{1}{4}\int_{-\infty}^{\infty} d\tau : [P^2 + X'^2] : \\
&+\frac{\sqrt{\pi}g_{st}}{12}\int_{-\infty}^{\infty} f_3(\tau) : [P_-(\tau)]^3 : +\frac{g_{st}}{8\sqrt{\pi}}\int_{-\infty}^{\infty} f_1(\tau)P_-(\tau)
\end{aligned}
\tag{13}
$$

where

$$
P_-(\tau) \equiv -i\int_0^\infty \frac{dk}{\sqrt{\pi}}\sqrt{k}[a(k)e^{ik\tau - i|k|t} - a^\dagger(k)e^{-ik\tau + i|k|t}]. \tag{14}
$$

Although the antisymmetric boundary condition is used to obtain the Hamiltonian (12), we will consider both of symmetric and antisymmetric boundary conditions. Actually the S matix we will evaluate is more similar to the previous result[4,5,6,7] in symmetric boundary condition

The LSZ reduction method is to obtain $S_{\beta\alpha} =< \beta\text{out} \mid \alpha\text{in} >=< \beta\text{in} \mid S \mid \alpha\text{in} >$ from the time-ordered products of quantum fields[11]. The $\mid \alpha\text{in} >$ and $< \beta\text{out} \mid$ are in- and out-states defined when interactions are turned off. Though Hamiltonian of our system is not invariant under Lorentz transformations, the free part ($\mathcal{H}(g_{st} = 0)$) is invariant under the transformations, which enables us to define the in- and out-states. Furthermore, it is not difficult to find out that the following LSZ reduction formula is true for both

of symmetric and antisymmetric boundary condition:

$$
\begin{aligned}
&< p_1 p_2 \cdots p_n \text{out} \mid q_1 \cdots q_m \text{in} > \\
&= (\frac{i}{\sqrt{Z}})^{m+n} \prod_{i=1}^{m} \int dt_i d\tau_i \prod_{j=1}^{n} \int dt'_j d\tau'_j e^{iq_i\tau_i - i|q_i|t_i} e^{-ip_j\tau'_j + i|p_j|t'_j} \\
&(\partial_{t_i}^2 - \partial_{\tau_i}^2)(\partial_{t'_j}^2 - \partial_{\tau'_j}^2) < T[X(t_1,\tau_1)\cdots X(t_n,\tau_n) X(t'_1,\tau'_1)\cdots X(t'_m,\tau'_m)] >,
\end{aligned} \tag{15}
$$

where Z is the constant for renormalization and is 1 for the theory of no divergence of loop expansion.

III. SYMMETRIC BOUNDARY CONDITION

In this section we will consider the case that X-field satisfies the symmetric boundary condition. In this boundary condition, the X-field can be written as

$$
X(t,\tau) = \int_0^\infty \frac{dk}{\sqrt{\pi k}} [a(k) e^{-i|k|t} + a^\dagger(k) e^{i|k|t}] \cos k\tau. \tag{16}
$$

To find the S matrix in LSZ reduction formalism, as discussed in previous section, time-ordered product of X-fields be calculated. For these calculations, the following formulas will be helpful

$$
\begin{aligned}
< T[X(t,\tau) P_-(t',\tau')] > &= -2 \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{dE}{2\pi} (E-k)\cos k\tau \\
&\quad e^{ik\tau'} e^{-iE(t-t')} \Delta^0(E,k), \qquad (17) \\
< T[P_-(t,\tau) P_-(t',\tau')] > &= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\frac{dE}{2\pi} 2ik e^{ik(\tau-\tau')} e^{-iE(t-t')} \Delta(E,k). \qquad (18)
\end{aligned}
$$

In Eqs.(17-18), Δ^0 and Δ are defined as

$$
\begin{aligned}
\Delta^0(E,k) &= \frac{1}{E^2 - k^2 + i\epsilon}, \qquad (19) \\
\Delta(E,k) &= \frac{1}{E - k + i\text{sgn}(k)\epsilon}. \qquad (20)
\end{aligned}
$$

It is tedious but straightforward applications of Wick's theorem to obtain the formula

$$
\begin{aligned}
& < T[X(t,\tau)X(t',\tau')] > \\
& =< T[X(t,\tau)X(t',\tau')] >^{(0)} \\
& \quad + \frac{ig_{st}^2}{(2\pi)^4}\int_{-\infty}^{\infty} dkdk'dEdk_adk_bk_ak_b\cos k\tau\cos k'\tau' e^{-iE(t-t')} \\
& \quad (E-k)g_3(k_a+k_b+k)(E-k')g_3(-k_a-k_b-k') \\
& \quad [\frac{\theta(k_a)\theta(k_b)}{E-k_a-k_b+i\epsilon} - \frac{\theta(-k_a)\theta(-k_b)}{E-k_a-k_b-i\epsilon}]\Delta^0(E,k)\Delta^0(E,k') \\
& \quad + \frac{ig_{st}^2}{2^5\pi^4}\int_{-\infty}^{\infty} dkdk'dEdk_ak_a\cos k\tau\cos k'\tau' e^{-iE(t-t')} \\
& \quad (E-k)(E-k')g_3(k-k'+k_a)g_1(-k_a)\Delta^0(E,k)\Delta(0,k_a)\Delta^0(E,k') \\
& \quad + O(g_{st}^4),
\end{aligned} \tag{21}
$$

where

$$
\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{otherwise.} \end{cases}
$$

The second and third term of Eq.(21) come from the one-loop and tadpole diagrams, respectively. While the superscript $^{(0)}$ denotes that it is order of g_{st}^0, $< T[X(t,\tau)X(t',\tau')] >^{(0)}$ does not contribute to S matrix.

Though Eq.(21) is rather complicated, S matrix element $< p_{out} \mid p_{in} >$ obtained by applying the LSZ reduction method (Eq.(15)) is simple.

$$
\begin{aligned}
& < p_{out} \mid p_{in} >= \\
& \frac{ig_{st}^2}{2^2\pi}\delta(|p_{in}|-|p_{out}|)|p_{in}| \\
& \quad \times \int_{-\infty}^{\infty} dk_adk_bk_ak_b[\frac{\theta(k_a)\theta(k_b)}{|p_{in}|+k_a+k_b-i\epsilon} - \frac{\theta(-k_a)\theta(-k_b)}{|p_{in}|+k_a+k_b+i\epsilon}] \\
& \quad g_3(k_a+k_b+|p_{in}|)g_3(-k_a-k_b-|p_{out}|)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{ig_{st}^2}{2^5\pi}\frac{\delta(|p_{in}|-|p_{out}|)}{|p_{in}|}\int_{-\infty}^{\infty}dk_a k_a\Delta(0,k_a)g_1(-k_a)\\
&\quad\int_{-\infty}^{\infty}dk'[\delta(p_{out}+k')+\delta(p_{out}-k')](|p_{out}|-k')\\
&\quad[(|p_{in}|-p_{in})g_3(k'+k_a-p_{in})+(|p_{in}|+p_{in})g_3(k'+k_a+p_{in})]\\
&+O(g_{st}^4)\\
&=\frac{ig_{st}^2}{2^2\pi}\delta(|p_{in}|-|p_{out}|)|p_{in}|\\
&\times\left\{\begin{array}{l}\int_{-\infty}^{\infty}dk_a dk_b k_a k_b[\frac{\theta(k_a)\theta(k_b)}{|p_{in}|+k_a+k_b-i\epsilon}-\frac{\theta(-k_a)\theta(-k_b)}{|p_{in}|+k_a+k_b+i\epsilon}]\\ \quad g_3(k_a+k_b+|p_{in}|))g_3(-k_a-k_b-|p_{in}|)\\ -\frac{1}{2}\int_{-\infty}^{\infty}dk_a k_a\frac{g_3(k_a)g_1(-k_a)}{-k_a+i\mathrm{sgn}(k_a)\epsilon}\end{array}\right.\\
&+O(g_{st}^4) \qquad (22)\\
&=\frac{ig_{st}^2}{24\pi}\delta(|p_{in}|-|p_{out}|)|p_{in}|\\
&\quad\times[i\pi|p_{in}|^3g_3^2(0)+3|p_{in}|^2\int_{-\infty}^{\infty}dsg_3(-s)g_3(s)+\int_{-\infty}^{\infty}ds\ s^2g_3(-s)g_3(s)\\
&\quad+3\int_{-\infty}^{\infty}dsg_1(-s)g_3(s)]+O(g_{st}^4) \qquad (23)\\
&=\frac{ig_{st}^2}{24}\delta(|p_{in}|-|p_{out}|)|p_{in}|\\
&\quad\times\left[\begin{array}{l}i|p_{in}|^3(\int_{-\infty}^{\infty}f_3(\tau)d\tau)^2+6|p_{in}|^2\int_{-\infty}^{\infty}(f_3(\tau))^2d\tau\\ +2\int_{-\infty}^{\infty}(\partial_\tau f_3(\tau))^2d\tau+6\int_{-\infty}^{\infty}f_3(\tau)f_1(\tau)d\tau\end{array}\right.\\
&+O(g_{st}^4). \qquad (24)
\end{aligned}
$$

To derive the Eq.(24) from Eq.(23), we use the convolution theorem of Fourier transforms

$$\int_{-\infty}^{\infty}g_a(s)g_b(-s)ds=2\pi\int_{-\infty}^{\infty}f_a(x)f_b(x)dx. \qquad (25)$$

The above derivation of $< p_{out} \mid p_{in} >$ shows a general property of S matrix. From the time-ordered product (17), one can find that the following

argument holds for any order of perturbation expansion:

$$
\begin{aligned}
&< X(t,\tau)\cdots > - < X(t,\tau)\cdots >^{(0)} \\
&\quad = \int dk dE \Delta^0(E,k) \cos k\tau (E-k) \int dt' e^{-iE(t-t')} \int d\tau' e^{ik\tau'} f_3(\tau') \\
&\qquad \times (\text{terms which do not contain } E,k,t,\tau). \qquad (26)
\end{aligned}
$$

If one consider a LSZ reduction which fixes k to an external momentum p, the contribution to S matrix of the time-ordered product in Eq.(26) is

$$
\begin{aligned}
&\frac{1}{2}\int dk dE[\delta(p+k)+\delta(p-k)]\delta(E\pm|p|)(E-k)g_3(k+\cdots)\int dt' e^{iEt'} \\
&\qquad \times (\text{terms which do not contain } p,E,k) \qquad (27) \\
&= \frac{1}{2}\int dk[\delta(p+k)+\delta(p-k)](\mp|p|-k)g_3(k+\cdots)\int dt' e^{\mp i|p|t'} \\
&\qquad \times (\text{terms which do not contain } p,k) \\
&= \mp|p|g_3(\pm|p|+\cdots)\int dt' e^{\mp i|p|t'} \\
&\qquad \times (\text{terms which do not contain } p). \qquad (28)
\end{aligned}
$$

Where the upper (lower) sign is for the case that the external energy $|p|$ is of in- (out-) state. The above argument proves that *the S matrix depends only on the energies of external particles in symmetric boundary condition.*

For the scattering process of three particles, we calculate the time-ordered product of three X-fields:

$$
\begin{aligned}
&< T[X(t_\alpha,\tau_\alpha)X(t_\beta,\tau_\beta)X(t_\gamma,\tau_\gamma)] > \\
&\quad = \frac{i\sqrt{\pi}g_{st}}{2^3\pi^5}\int_{-\infty}^{\infty} dk_\alpha dE_\alpha dk_\beta dE_\beta dk_\gamma dE_\gamma \delta(E_\alpha+E_\beta+E_\gamma) e^{-iE_\alpha t_\alpha - iE_\beta t_\beta - iE_\gamma t_\gamma} \\
&\qquad \Delta^0(E_\alpha,k_\alpha)\Delta^0(E_\beta,k_\beta)\Delta^0(E_\gamma,k_\gamma)\cos k_\alpha\tau_\alpha \cos k_\beta\tau_\beta \cos k_\gamma\tau_\gamma \\
&\qquad (E_\alpha-k_\alpha)(E_\beta-k_\beta)(E_\gamma-k_\gamma)g_3(k_\alpha+k_\beta+k_\gamma)
\end{aligned}
$$

$$+ O(g_{st}^3). \tag{29}$$

By applying the formula (15) to this time-ordered product one can find

$$\begin{aligned} &< p_\gamma \text{out} \mid p_\alpha p_\beta \text{in} > \\ &\quad = -g_{st}\delta(|p_\alpha| + |p_\beta| - |p_\gamma|)\sqrt{|p_\alpha p_\beta p_\gamma|}g_3(0) + O(g_{st}^3) \\ &\quad = -g_{st}\delta(|p_\alpha| + |p_\beta| - |p_\gamma|)\sqrt{|p_\alpha p_\beta p_\gamma|}\int_{-\infty}^{\infty} f_3(\tau)d\tau + O(g_{st}^3). \end{aligned} \tag{30}$$

One of the remarkable properties of LSZ reduction is the symmetry between in- and out-states [15]. If we replace replace energy and momentum ($|p|$, p) of an in-state particle by ($-|p|$, $-p$) besides square root of product of external particles' energies in the formula (15), then the formula gives the S matrix for which the in-state particle is replaced by a out-state particle of ($|p|$, p). From this symmetry and Eq.(30), one can directly find, for example, that

$$< p_\beta p_\gamma \text{out} \mid p_\alpha \text{in} >= g_{st}\delta(|p_\alpha| - |p_\beta| - |p_\gamma|)\sqrt{|p_\alpha p_\beta p_\gamma|}\int_{-\infty}^{\infty} f_3(\tau)d\tau + O(g_{st}^3). \tag{31}$$

Again, by applying the Wick's theorem, one finds

$$\begin{aligned} &< T[X(t_\alpha,\tau_\alpha)X(t_\beta,\tau_\beta)X(t_\gamma,\tau_\gamma)X(t_\delta,\tau_\delta)] > \\ &=< T[X(t_\alpha,\tau_\alpha)X(t_\beta,\tau_\beta)X(t_\gamma,\tau_\gamma)X(t_\delta,\tau_\delta)] >^{(0)} \\ &\quad - \frac{ig_{st}^2}{2^5\pi^7}\int_{-\infty}^{\infty} dk_\alpha dE_\alpha dk_\beta dE_\beta dk_\gamma dE_\gamma dk_\delta dE_\delta \delta(E_\alpha + E_\beta + E_\gamma + E_\delta) \\ &\quad e^{-iE_\alpha t_\alpha - iE_\beta t_\beta - iE_\gamma t_\gamma - iE_\delta t_\delta}\Delta^0(E_\alpha, k_\alpha)\Delta^0(E_\beta, k_\beta)\Delta^0(E_\gamma, k_\gamma)\Delta^0(E_\delta, k_\delta) \\ &\quad \cos k_\alpha\tau_\alpha \cos k_\beta\tau_\beta \cos k_\gamma\tau_\gamma \cos k_\delta\tau_\delta (E_\alpha + k_\alpha)(E_\beta + k_\beta)(E_\gamma + k_\gamma)(E_\delta + k_\delta) \\ &\quad \int_{-\infty}^{\infty} dkk \left[\begin{array}{l} \Delta(E_\alpha + E_\beta, k)g_3(-k_\alpha - k_\beta + k)g_3(-k_\gamma - k_\delta - k) \\ +(\beta \leftrightarrow \gamma) + (\beta \leftrightarrow \delta) \end{array} \right. \\ &\quad + O(g_{st}^4). \end{aligned} \tag{32}$$

which, through LSZ reduction method, yields

$$
\begin{aligned}
&< p_\gamma p_\delta \text{out}|p_\alpha p_\beta \text{in} > \\
&= \frac{ig_{st}^2}{2\pi}\delta(|p_\alpha|+|p_\beta|-|p_\gamma|-|p_\delta|)\sqrt{|p_\alpha p_\beta p_\gamma p_\delta|} \\
&\times \left[\begin{array}{l} 3\int_{-\infty}^{\infty} g_3(-s)g_3(s)ds \\ +i\pi g_3^2(0)[|p_\alpha|+|p_\beta|+|(|p_\alpha|-|p_\gamma|)|+|(|p_\alpha|-|p_\delta|)|] \end{array} \right. \\
&+ O(g_{st}^4) \qquad (33) \\
&= \frac{ig_{st}^2}{2}\delta(|p_\alpha|+|p_\beta|-|p_\gamma|-|p_\delta|)\sqrt{|p_\alpha p_\beta p_\gamma p_\delta|} \\
&\times \left[\begin{array}{l} 6\int_{-\infty}^{\infty}(f_3(\tau))^2 d\tau \\ +i(\int_{-\infty}^{\infty} f_3(\tau)d\tau)^2[|p_\alpha|+|p_\beta|+|(|p_\alpha|-|p_\gamma|)|+|(|p_\alpha|-|p_\delta|)|] \end{array} \right. \\
&+ O(g_{st}^4). \qquad (34)
\end{aligned}
$$

IV. ANTISYMMETRIC BOUNDARY CONDITION

For the antisymmetric boundary condition, the X-field can be written as

$$X = i\int_0^\infty \frac{dk}{\sqrt{\pi k}}[a(k)e^{-i|k|t} - a^\dagger(k)e^{-i|k|t}]\sin k\tau, \qquad (35)$$

which gives the time-ordered product

$$
\begin{aligned}
< T[X(t,\tau)P_-(t',\tau')] > \;&=\; 2i\int_{-\infty}^{\infty}\frac{dk}{2\pi}\frac{dE}{2\pi}(E-k)\sin k\tau \\
&\qquad e^{ik\tau'}e^{-iE(t-t')}\Delta^0(E,k). \qquad (36)
\end{aligned}
$$

Since the above time-ordered product can be obtained from that of symmetric boundary condition (Eq.(17)) by replacing $\cos k\tau$ with $\frac{1}{i}\sin k\tau$, the same is true for any part of time-ordered product $< T[XX\cdots X] >$ of order of g_{st}^n $(n \geq 1)$. ($< T[XX] >^{(0)}$ in antisymmetric boundary condition also does

not contribute to S matrix as in the symmetric boundary condition.) As an example, $< T[X(t_\alpha,\tau_\alpha)X(t_\beta,\tau_\beta)X(t_\gamma,\tau_\gamma)] >$ of antisymmetric boundary condition can be directly obtained from Eq.(29):

$$
\begin{aligned}
&< T[X(t_\alpha,\tau_\alpha)X(t_\beta,\tau_\beta)X(t_\gamma,\tau_\gamma)] > \\
&\quad = -\frac{\sqrt{\pi}g_{st}}{2^3\pi^5}\int_{-\infty}^{\infty} dk_\alpha dE_\alpha dk_\beta dE_\beta dk_\gamma dE_\gamma \delta(E_\alpha+E_\beta+E_\gamma) e^{-iE_\alpha t_\alpha - iE_\beta t_\beta - iE_\gamma t_\gamma} \\
&\qquad \Delta^0(E_\alpha,k_\alpha)\Delta^0(E_\beta,k_\beta)\Delta^0(E_\gamma,k_\gamma)\sin k_\alpha\tau_\alpha \sin k_\beta\tau_\beta \sin k_\gamma\tau_\gamma \\
&\qquad (E_\alpha-k_\alpha)(E_\beta-k_\beta)(E_\gamma-k_\gamma) g_3(k_\alpha+k_\beta+k_\gamma) \\
&\quad + O(g_{st}^3).
\end{aligned}
\tag{37}
$$

As in the symmetric boundary condition, therefore,

$$
\begin{aligned}
&< X(t,\tau)\cdots > - < X(t,\tau)\cdots >^{(0)} \\
&\quad = \int dk dE \Delta^0(E,k)\frac{\sin k\tau}{i}(E-k)\int dt' e^{-iE(t-t')}\int d\tau' e^{ik\tau'} f_3(\tau') \\
&\qquad \times (\text{terms which do not contain } E,k,t,\tau).
\end{aligned}
\tag{38}
$$

And the contribution to S matrix will be

$$
\begin{aligned}
&\mp\frac{1}{2}\int dk dE[\delta(p+k)-\delta(p-k)]\delta(E\pm|p|)(E-k)g_3(k+\cdots)\int dt' e^{iEt'} \\
&\qquad \times (\text{terms which do not contain } p,E,k) \qquad (39) \\
&= \mp\frac{1}{2}\int dk[\delta(p+k)-\delta(p-k)](\mp|p|-k)g_3(k+\cdots)\int dt' e^{\mp i|p|t'} \\
&\qquad (\text{terms which do not contain } p,k) \\
&= \mp p g_3(\pm|p|+\cdots)\int dt' e^{\mp i|p|t'} \\
&\qquad \times (\text{terms which do not contain } p).
\end{aligned}
\tag{40}
$$

where the upper(lower) sign is for the case that the external momentum p is of in-(out-)state.

Therefore, *the S matrix in antisymmetric boundary condition is proportional to the momenta of external particles,* while it is proportional to energies in symmetric boundary condition. Except these parts, S matrix are same in both cases and described by the energies of external particles.

It is easy to find the following formulas from the results in previous section and the symmetries between in- and out- states in LSZ reduction method;

$$
\begin{aligned}
&< p_{out} \mid p_{in} > \\
&= \frac{ig_{st}^2}{24}\delta(|p_{in}|-|p_{out}|)\frac{p_{in}p_{out}}{|p_{in}|} \\
&\quad \times \left[\begin{array}{l} i|p_{in}|^3(\int_{-\infty}^{\infty} f_3(\tau)d\tau)^2 + 6|p_{in}|^2\int_{-\infty}^{\infty}(f_3(\tau))^2 d\tau \\ +2\int_{-\infty}^{\infty}(\partial_\tau f_3(\tau))^2 d\tau + 6\int_{-\infty}^{\infty} f_3(\tau)f_1(\tau)d\tau \end{array} \right. \\
&\quad +O(g_{st}^4),
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
&< p_\delta \mathrm{out} \mid p_\alpha p_\beta p_\gamma \mathrm{in} > \\
&= -\frac{ig_{st}^2}{2}\delta(|p_\alpha|+|p_\beta|+|p_\gamma|-|p_\delta|)\frac{p_\alpha p_\beta p_\gamma p_\delta}{\sqrt{|p_\alpha p_\beta p_\gamma p_\delta|}} \\
&\quad \times \left[\begin{array}{l} 6\int_{-\infty}^{\infty}(f_3(\tau))^2 d\tau \\ +i(\int_{-\infty}^{\infty} f_3(\tau)d\tau)^2[2|p_\alpha|+|p_\beta|+|p_\gamma|+|(|p_\alpha|-|p_\delta|)|] \end{array} \right. \\
&\quad + O(g_{st}^4).
\end{aligned} \tag{42}
$$

V. APPLICATIONS

For the inverted harmonic oscillator potential $U(\lambda) = -\frac{\lambda^2}{2}$, v can be written as

$$v(\tau) = \sqrt{2|\mu|}\cosh\tau.$$

For this case, Eqs. (23), (30), (33) in symmetric boundary condition are essentially same to those of Ref.[7] except factors, square root of the product

of external energies, and the results also agree with those in Ref.[1,2]. So one can easily find S matrices. For examples, in antisymmetric boundary condition, one can find

$$\begin{aligned} < p_{out} \mid p_{in} > &= \frac{ig_{st}^2}{24}\delta(|p_{in}| - |p_{out}|)\frac{p_{in}p_{out}}{|p_{in}|}(i|p_{in}|^3 + 2|p_{in}|^2 + 1) \\ &\quad + O(g_{st}^4), \end{aligned} \tag{43}$$

$$\begin{aligned} < p_\gamma p_\delta \text{out} \mid p_\alpha p_\beta \text{in} >= -\frac{g_{st}^2}{2}\delta(|p_\alpha| + |p_\beta| - |p_\gamma| - |p_\delta|)\frac{p_\alpha p_\beta p_\gamma p_\delta}{\sqrt{|p_\alpha p_\beta p_\gamma p_\delta|}} \\ \times [|p_\alpha| + |p_\beta| + |(|p_\alpha| - |p_\gamma|)| + |(|p_\alpha| - |p_\delta|)| - 2i] + O(g_{st}^4). \end{aligned} \tag{44}$$

Thanks to the convolution theorem, however, the following formula may be enough to derive Eqs.(43-44)

$$\int_0^\infty \frac{dx}{\cosh^{2n}(\beta x)} = \frac{2^{2(n-1)}[(n-1)!]^2}{(2n-1)!\beta}. \tag{45}$$

As a new application, we consider a potential

$$U(\lambda) = -\frac{\lambda^2}{2} - \frac{\mu^2}{2\lambda^2} \tag{46}$$

with zero fermi energy, which was proposed in Ref.[16] and conjectured to describe a "naked singularity"[12] (see also Ref.[17]). After some algebras, one can find that

$$v(\tau) = \sqrt{|\mu|}\frac{\cosh 2\tau}{\sqrt{|\sinh 2\tau|}}, \tag{47}$$

$$f_1(\tau) = \frac{1}{2|\sinh 2\tau|} + \frac{4|\sinh 2\tau|}{3\cosh^2 2\tau} - 2\frac{|\sinh^3 2\tau|}{\cosh^4 2\tau}, \tag{48}$$

and

$$f_3(\tau) = \frac{|\sinh 2\tau|}{\cosh^2 2\tau}. \tag{49}$$

Making use of general expressions in the preceding two sections and Eq.(45), it is easy to find the S matrix elements to the order of g_{st}^2. For example, in symmetric boundary condition,

$$
\begin{aligned}
< p_{out} \mid p_{in} > \quad &= \quad \frac{i g_{st}^2}{24} \delta(|p_{in}| - |p_{out}|) |p_{in}| (\frac{i}{9} |p|^3 + 2|p_{in}|^2 + 7) \\
&\qquad + O(g_{st}^4), \qquad (50)
\end{aligned}
$$

$$
< p_\gamma \text{out} \mid p_\alpha p_\beta \text{in} >= -\frac{g_{st}}{3} \sqrt{|p_\alpha p_\beta p_\gamma|} \delta(|p_\alpha| + |p_\beta| - |p_\gamma|) + O(g_{st}^3). \qquad (51)
$$

$$
\begin{aligned}
< p_\gamma p_\delta \text{out} \mid p_\alpha p_\beta \text{in} > &= -\frac{g_{st}^2}{18} \delta(|p_\alpha| + |p_\beta| - |p_\gamma| - |p_\delta|) \sqrt{|p_\alpha p_\beta p_\gamma p_\delta|} \\
&\times [|p_\alpha| + |p_\beta| + |(|p_\alpha| - |p_\gamma|)| + |(|p_\alpha| - |p_\delta|)| - 18i] + O(g_{st}^4). \quad (52)
\end{aligned}
$$

The fact that $< p_\gamma \text{out} | p_\alpha p_\beta \text{in} > \neq 0$ has been noted in Ref.[12].

VI. CONCLUSION

We have studied the S matrix of $\mu < 0$ collective field theory by applying the LSZ reduction method. We consider the spatially symmetric and antisymmetric boundary conditions. In the antisymmetric boundary condition, the S matrices are proportional to the momenta of external particles, while in symmetric boundary condition they are proportional to the energies. Besides these factors, the S matrices are same in both boundary conditions. In the sense that the S matrices are described by natural variables (momenta and/or energies), both of the boundary conditions may be thought to be physically acceptable.

To the order of g_{st}^2, we find simple formulas of S matrices in terms of velocity. In Ref.[12] (see also Ref.[17]), it has been hypothesized that for the

theory of black hole the S matrix of odd particles must vanish. This is not possible in $\mu < 0$ case since the S matrix of three external particles proportional to $\int_{-\infty}^{\infty} \frac{1}{v^2(\tau)} d\tau$ and v is always larger than 0. Furthermore, it mostly looks like that the $\mu < 0$ S matrices of different potentials have universal form when they are expanded in terms of energies of external particles. The effects of different potentials would appear only in the coefficients of the expansions.

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